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# SOME ANALYTICAL ASPECTS OF THE PEAKEDNESS CONCEPT

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# Some analytical aspects of the peakedness concept

by

E.A. van Doorn

ABSTRACT

In teletraffic engineering one commonly uses as a second order characterization of traffic its peakedness factor, which is defined as the variance-to-mean ratio for the trunk occupancy distribution resulting when the traffic is offered to an infinite trunk group. Assuming renewal input streams and exponentially distributed holding times, Pearce [21] and others have given representations and bounds for the peakedness factors of primary and secondary traffic streams. In this paper some of these results will be generalized and some new results in this vein will be obtained.

KEY WORDS & PHRASES: *teletraffic theory, peakedness factor*

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## 1. INTRODUCTION

The customary basic model in teletraffic theory is that of a finite or infinite trunk group to which a renewal stream of calls with exponentially distributed holding times is offered. Considering that most traffic streams in a network will result from superposition of other streams (so that they are not in general renewal), it is of some interest to relax the renewal assumption and to investigate whether a more general setting yields to analysis. Therefore, we start out to formally define traffic as a stochastic marked point process, i.e., a sequence $\{(t_i,h_i): -\infty<i<\infty\}$, where the points $t_i$ correspond to arrivals of calls and the marks $h_i$ are the holding times associated with these calls. Throughout this paper we will assume that only one call can arrive at a particular point in time. Further, the marked point process is always supposed to be stationary and metrically transitive. A formal definition of a marked point process and the associated concepts may be found in, e.g., [11], [12], [14] and [19].

Traffic engineers tend to be interested in the stationary distribution of the number of busy trunks in an infinite of finite trunk group which is induced by a particular traffic stream. Indeed, traffic is often defined by the distribution it induces on an infinite trunk group. This is justified when the point process of arriving calls is a renewal process, and the associated holding times constitute a sequence of independent and exponentially distributed random variables with known, common mean, since then there is a one-to-one correspondence between the interarrival time distribution and the trunk occupancy distribution on the infinite group, as was shown by Wallin (private communication*)). In general, however, this is probably not the case. Anyhow, from a practical point of view even this distribution is unmanageable, so that interest centers on a few of its moments, usually the first two. Rather than, for instance, mean and variance, one uses mean and variance-to-mean ratio of the number of busy trunks on an infinite trunk group as traffic characteristics. Indeed, these quantities are meant when one speaks of the *mean* and *peakedness* (*factor*), respectively, of a particular traffic stream.

*) The problem essentially concerns the question of whether the sequence $\{\phi(m): m=0,1,\ldots\}$, where $\phi$ is the Laplace-Stieltjes transform of the interarrival time distribution F, may be generated by more than one distribution. However, the problem of finding F given $\{\phi(m)\}$ can be formulated as a Hausdorff moment problem, which as at most one solution.

Note that carried traffic, i.e., that part of a traffic stream which consists of the calls (points in time) which are effectively served on a finite trunk group, and the associated holding times, induces the same distribution on an infinite trunk group as the original traffic stream on the finite carrier group. Thus mean and peakedness of carried traffic are equal to (and will be identified with) mean and variance-to-mean ratio, respectively, of the trunk occupancy distribution on the finite carrier group. We remark, however, that in the context of carried traffic some authors use another definition of peakedness (see the last paragraph of Section III).

There are indications that a two-moment characterization of traffic is adequate in practice, provided the holding times are independent of the interarrival times ([30], see also [17]). But when the independence assumption is no longer valid (as is generally the case with carried traffic), it is doubtful whether mean and peakedness are sufficiently accurate in describing a traffic stream. However, we shall not be concerned with this problem in this paper, where we restrict ourselves to a theoretical analysis of mean and peakedness.

The important theoretical questions of whether a traffic stream as defined above induces a unique, stationary distribution of busy trunks on a finite or infinite trunk group was answered in the affirmative by Franken and Kerstan [13] and Franken [9], at least in the cases that we will consider (see [2], [11], [12], [14] and [19] for related and more general results). The above reservation refers to the fact that we assume throughout that holding times are mutually independent random variables with an exponential distribution of mean $1/\mu$. Further, we shall only consider *free traffic* and secondary forms thereof (carried traffic and overflow traffic), where free traffic is defined to be traffic where the holding times are independent of the point process of arriving calls.

The organization of this paper is as follows. In Section 2 we derive a representation formula for the peakedness factor of free traffic. This formula is subsequently used to produce a lower bound for the peakedness factor. In Sections 3 and 4 the peakedness factors of carried traffic and overflow traffic, respectively, will be studied and related to the peakedness factor of the associated (free) offered traffic. Section 5 contains some folklore results on the peakedness of *renewal traffic,* i.e., traffic where the point process of arriving calls constitutes a renewal process.

## 2. FREE TRAFFIC

We start off with some notation. Consider a free traffic stream as defined in the previous section and suppose that it is offered to an infinite trunk group. The stationary distribution of the number of busy trunks at an arbitrary moment will be denoted by $\{p(n)\}$. We are also interested in the trunk occupancy distribution just prior to the arrival of a call, which will be denoted by $\{p'(n)\}$. Finally $p^*(n)$ stands for the probability of n busy trunks just after the departure (end) of a call. The factorial moments of these distributions will be denoted by $M_{(k)}$, $M'_{(k)}$ and $M^*_{(k)}$, respectively, i.e., $M_{(0)} = 1$ and

$$M_{(k)} = \sum_{j=k}^{\infty} j(j-1)\dots(j-k+1)p(j), \quad k = 1,2,\dots,$$

etc. . The first (factorial) moments will be denoted by M, M' and $M^*$, instead of $M_{(1)}$, $M'_{(1)}$ and $M^*_{(1)}$, respectively.

We shall now give the relations that exist between the above distributions. First, it is well known (see [10], [11] or Section 5.2 of [4]) that

$$(1) \qquad p'(n) = p^*(n), \quad n = 0,1,\dots .$$

Further, we have the important relation

$$(2) \qquad Ap'(n) = (n+1)p(n+1), \quad n = 0,1,\dots$$

(see [10], [11]), where A is the *traffic intensity,* which is defined as the intensity of the point process of arriving calls $\lambda$, say, times the mean holding time $1/\mu$. From (2) one readily deduces the relation that exists between the factorial moments $M_{(k)}$ and $M'_{(k)}$, viz.,

$$(3) \qquad AM'_{(k)} = M_{(k+1)}, \quad k = 0,1,\dots,$$

a result which was first given by Franken and Kerstan [13]. Note that formula (3) with $k = 0$ ($A=M$) is implied by Little's formula.

The peakedness factor $z$ of free traffic is defined as the variance-to-mean ratio of the distribution $\{p(n)\}$, i.e.,

$$(4) \qquad z = V/M,$$

where

$$(5) \qquad V = M_{(2)} + M - M^2.$$

By (3) we have $M_{(2)} = AM'$ (and $M=A$), so that the next theorem emerges.

<u>THEOREM 1</u>. *For the mean* $M$ *and peakedness* $z$ *of free traffic with intensity* $A$ *one has* $M = A$ *and*

$$(6) \qquad z = 1 + M' - M.$$

This theorem was first observed by Descloux [6] and later by Pearce and Potter [23] for renewal traffic. Heffes and Holtzman [16] proved its validity for a traffic stream whose point process is that of carried traffic of renewal offered traffic, but whose original holding times are replaced by new ones, which are then independent of the point process of calls (*freed carried traffic*).

Note that the dichotomy $z > 1$ (*peaked traffic*) vs. $z < 1$ (*smooth traffic*) has an interesting interpretation in the form $M' > M$ vs. $M' < M$.

A traffic stream will be called *regular* when the point process of arriving calls is a renewal process with constant interarrival times. The quantities pertaining to regular traffic will be indexed by the letter R. We now cite an important result of Franken and Kerstan [13], to the effect that for any traffic stream

$$(7) \qquad M'_{(k)}(A) \geqq M'_{R(k)}(A),$$

where we have indicated dependence on the traffic intensity A. In particular one has

$$M'(A) \geqq M'_R(A), \tag{8}$$

so that (6) yields

$$z(A) \geqq z_R(A). \tag{9}$$

This result is in accordance with the usual interpretation of peakedness as a measure of variability of the input stream. To obtain an explicit lower bound for $z(A)$ we must calculate $z_R(A)$. The result of this calculation is given, e.g., in formula (39) of Section 5. Subsequent substitution in (9) yields the following theorem.

THEOREM 2. *The peakedness factor* $z(A)$ *of free traffic with intensity* $A$ *satisfies*

$$z(A) \geqq z_R(A) = \{1-\exp(-1/A)\}^{-1} - A. \tag{10}$$

The inequality in this theorem is essentially a relation between variances, in which form it was stated already by Franken and Kerstan [13]. The validity of the theorem when one restricts oneself to renewal traffic was observed by Kuczura [18] and Pearce [21].

It is easily seen that $z_R(A)$ is a decreasing function with the values $z_R(0) = 1$ and $z_R(\infty) = \frac{1}{2}$. Hence we have the next corollary.

COROLLARY. *The peakedness factor* $z$ *of any stream of free traffic statisfies*

$$z > \tfrac{1}{2}. \tag{11}$$

The latter inequality has an interesting interpretation. Namely, let $\hat{M}$ denote the expected number of busy trunks on the infinite trunk group immediately before an event (either arrival or departure). Then, since each arrival corresponds to a departure and vice versa,

$$\hat{M} = \tfrac{1}{2}\{M' + (M^*+1)\}.$$

(Of course, $\hat{M}$ is also equal to the number of busy trunks immediately after an event.) From (1) we see that $M' = M^*$. Hence

$$\hat{M} = M' + \tfrac{1}{2}. \tag{12}$$

We now obtain

$$\hat{M} > M, \tag{13}$$

since, by (6) and (11), $M' + \frac{1}{2} = z + M - \frac{1}{2} > M$. Thus the expected number of busy trunks at an arbitrary moment is always smaller than at a moment where the system changes state. (Here "at" can have the interpretation "just prior to" as well as "just after"; in what follows, however, we shall always mean the former in the case of ambiguity.)

As a final remark in this section we mention the fact that peakedness of free traffic with fixed intensity can be made arbitrary large. An example is provided by renewal traffic with an interarrival distribution which has a mass p at $x_0 > 0$ and a mass $1-p$ at $x_1 > x_0$, where $p = (x_1-1/\lambda)/(x_1-x_0)$. The well-known formula (33) for the peakedness factor z of renewal traffic readily yields that z exceeds any bound by choosing $x_0$ sufficiently small and $x_1$ sufficiently large. (Incidentally, this is the example used by Beneš [1] (see also Pearce [22]) to show that when traffic with fixed intensity is offered to a finite group of fixed size, the blocking probability can be arbitrarily close to unity.)

## 3. CARRIED TRAFFIC

Consider a marked point process representing a stream of free traffic with traffic intensity $A = \lambda/\mu$, which we will designate as offered traffic. When offered to a finite trunk group of size N, this stream is split in two parts: carried traffic and overflow traffic. This section will be concerned with the former. A subindex ca will distinguish quantities pertaining to carried traffic from those belonging to the offered traffic. We recall that an analysis of the trunk occupancy distribution on an infinite trunk group offered this carried traffic amounts to a study of the trunk occupancy distribution on the trunk group of size N to which the original free traffic is offered. This latter distribution (at an arbitrary moment) will be denoted by $\{p_{ca}(n)\}$. The distribution at an arrival moment (of the offered stream) is dentoted by $\{p'_{ca}(n)\}$. The two distributions are related as

$$(14)\qquad \begin{aligned} Ap'_{ca}(n) &= (n+1)p_{ca}(n+1),\ n = 0,\dots,N-1 \\ Ap'_{ca}(N) &= A - \sum_{n=0}^{N-1} (n+1)p_{ca}(n+1) \end{aligned}$$

([9]-[11]; see [24] for more general results). It is easy now to deduce from (14) that the factorial moments for these distributions are related as

$$(15)\qquad AM'_{ca(k)} = M_{ca(k+1)} + \frac{N!}{(N-k)!}(A-M_{ca}),\ k = 0,\dots,N-1,$$

where the notation should be clear. The last equality of (14) can be written as

$$(16)\qquad M_{ca} = A(1-B),$$

where

$$B = p'_{ca}(N) = M'_{ca(N)}/N!,$$

i.e., B is the blocking probability or *call congestion*. Formulas (15) and (16) imply that $M_{ca(2)} = A(M'_{ca}-NB)$, so that the peakedness factor $z_{ca} = V_{ca}/M_{ca} = (M_{ca(2)}+M_{ca}-M^2_{ca})/M_{ca}$ of carried traffic is given by

$$(17)\qquad z_{ca} = 1 - M_{ca} + (M'_{ca}-NB)/(1-B).$$

Clearly, $M'_{ca}$, the expected number of busy trunks in the infinite group at an arrival moment, equals B times N, the number of busy trunks on the finite group at an overflow moment, plus 1-B times the expected number of busy trunks at a moment where a call is accepted on that group. It follows that the latter quantity $M_{ca}^{ca}$, say, is given by

$$(18) \qquad M_{ca}^{ca} = (M'_{ca}-NB)/(1-B).$$

Substitution of (18) in (17) gives us the analogue of (6) for carried traffic

$$(19) \qquad z_{ca} = 1 + M_{ca}^{ca} - M'_{ca}.$$

REMARK. This formula was observed by Descloux [6] for renewal offered traffic. Descloux also considered a model where a finite waiting room is available for calls that arrive when the trunk group is full. His formula (26) is not true then, but it may be shown that (19) is valid for this model too, even when the offered traffic is non-renewal. Again, $M_{ca}^{ca}$ should be interpreted as the expected number of busy trunks in the group at a moment where a call is accepted on the group (this includes moments at which a call is shifted from a waiting position to a trunk, i.e., moments at which a call finishes while the waiting room is not empty.)

To obtain more explicit results on the peakedness factor of carried traffic we must impose an additional condition on the offered traffic.

LEMMA 1. *If the offered traffic is renewal then* $M'_{ca} = (1-B)M'$.

PROOF. Let $M'_{ov}$ denote the expected number of busy trunks on an infinite overflow group at an arrival moment (of the offered stream). We must show that $M'_{ov}/M'_{ca} = B/(1-B)$, since $M' = M'_{ca} + M'_{ov}$. Now, because of the renewal character of the input stream and the exponentially distributed holding times, the ratio of the expected number of busy trunks on the overflow group and on the finite group just prior to an arrival will be equal to that ratio just after an arrival. It follows that this ratio must be $B/(1-B)$, since an arriving call will occupy a trunk on the overflow group with probability $B$ and on the finite carrier group with probability $1-B$. □

The relation (17) combined with (16) and the above lemma immediately yields the following theorem, which seems not to have been published before, although an alternative proof can readily be given by using results that are available in the literature.

THEOREM 3. *When renewal traffic with intensity* A *and peakedness factor* z *is offered to a group of* N *trunks, then the peakedness factor* $z_{ca}$ *of the traffic carried by that group is given by*

$$(20) \qquad z_{ca} = z + AB - NB/(1-B),$$

*where* B *is the blocking probability experienced by the offered traffic.*

REMARK. Formula (20) may be formulated alternatively as

$$(20') \qquad z_{ca} = z - (A-M_{ca})(N-M_{ca})/M_{ca}.$$

Theorem 3 is well known for offered Poisson traffic (cf. p. 498 of [30]). For overflow traffic of Poisson traffic, the particular form (20) seems to have been observed by Heffes (cf. p. 819 of [15]).

Evidently, $N > M_{ca} = A(1-B)$, so that we have the following corollary of Theorem 3.

COROLLARY. *For the peakedness factors* $z_{ca}$ *and* z *of traffic carried on a finite group and the associated offered renewal traffic, respectively, one has the relation*

$$(21) \qquad z_{ca} < z.$$

An example may be constructed showing that Theorem 3 is not generally valid for offered non-renewal traffic (cf. [29]). The hitch is in the proof of Lemma 1 where we have used the independence of the trunk occupancy at an arrival moment and the time until the next arrival. Of course, this does not preclude (21) to be valid for non-renewal traffic. We conjecture that the latter is true indeed.

REMARK. Pearce [21] claims for offered renewal traffic that $z_{ca} < \max(1,z)$, which is correct, but also that $z_{ca}$ may be larger than $z$, which is contradictory to the above corollary.

The question arises whether the extremal property of regular traffic given in (7) carries over to quantities related to a finite carrier group. One would suspect, for instance, that the blocking experienced by calls from regular traffic offered to a trunk group of size N should be smaller than the blocking experienced by calls from any other type of free traffic with the same intensity. This was shown to be true for offered renewal traffic by Beneš [1] (see also [25] and Section 7.5 of [26]). However, for the more general class of free traffic as defined in this paper, the conjecture has only been validated for N = 1 by Franken [9] and for N = 2 by Fleischmann [8].

Inequalities involving the peakedness factor of carried regular traffic are not known with the exception of the case N = 1. By (16) and (19) one then has $z_{ca} = z_{ca}(A) = 1 - A(1-B)$, so that, in view of Franken's result, $z_{ca}(A) \geqq z_{R,ca}(A)$.

In closing this section we remark that Heffes and Holtzman [16] define peakedness of carried traffic as (in our terminology) the peakedness factor of freed carried traffic, i.e., carried traffic where the calls are provided with new independent holding times of the same exponential distribution as the old ones. Indeed, one can argue that this definition of peakedness serves better the purpose of describing the variability of the input stream of calls than the usual definition. We will not digress at this point, however, and just give a conjecture involving the peakedness factor $\hat{z}_{ca}$ of freed carried traffic and the peakedness factors $z_{ca}$ and $z$ of the associated traffics, to the effect that $z_{ca} < \hat{z}_{ca} < z$.

## 4. OVERFLOW TRAFFIC

As in the previous section we consider a stream of free traffic with traffic intensity $A = \lambda/\mu$, and suppose that it is offered to a group of N trunks. Those calls which are not carried on the finite group and their associated holding times constitute the overflow traffic. Clearly, overflow traffic is free traffic, so that we can invoke Theorem 1 to conclude that

(22) $$z_{ov} = 1 + M_{ov}^{ov} - M_{ov},$$

where a subindex ov refers to the infinite overflow group and a superindex ov to the fact that the pertinent mean is defined at overflow moments.

REMARK. The existence and uniqueness of a stationary trunk occupancy distribution (at an arbitrary moment) on the infinite overflow group follows readily from the results mentioned in the introduction. Consequently (see, e.g., [11], [12], [19]), there is also a unique stationary distribution at overflow moments.

By Little's formula, or, alternatively, by (16) and the fact that $A = M = M_{ca} + M_{ov}$, we have

(23) $$M_{ov} = AB,$$

B being the calll congestion experienced by the offered traffic, so that

(24) $$z_{ov} = 1 + M_{ov}^{ov} - AB.$$

Again we cannot get much further unless we impose the additional condition of renewal input. Doing this, we can cite a result of Pearce [21] stating that

(25) $$N = 1 \Rightarrow z_{ov} \geqq 1 - B + Bz,$$

with equality subsisting only if the offered traffic is regular. This result can be generalized as follows.

THEOREM 4. *Let renewal traffic with peakedness factor* z *be offered to a finite trunk group of size* N. *In terms of* z *and the blocking probability* B *experienced by the offered traffic, the peakedness factor* $z_{ov}$ *of the overflow traffic satisfies*

(26) $$z_{ov} \geqq 1 - B + Bz,$$

*with equality subsisting only if* $N = 1$ *and the offered traffic is regular.*

PROOF. Let the trunks in the finite group be numbered $1,2,\dots,N$ and suppose that an accepted call is carried on the lowest numbered free trunk. Let $z(i)$ denote the peakedness factor of the traffic which is offered to trunk $i + 1$, so that in particular $z(0) = z$ and $z(N) = z_{ov}$. Finally, let $B_i$ denote the blocking probability experienced on trunk i by the traffic offered to this trunk, so that, clearly,

$$(27) \qquad B = B_1 B_2 \dots B_N.$$

By (25) and the well-known result of Palm [20] that overflow traffic is renewal when the offered traffic is renewal, we then have

$$(28) \qquad z(i) \geqq 1 - B_i + B_i z(i-1), \; i = 1,\dots,N.$$

Consequently,

$$z_{ov} = z(N) \geqq 1 - (1-z(0))\prod_1^N B_i = 1 - B + Bz.$$

The remaining part is evident. □

Theorem 4 is readily seen to imply Pearce's [21] result for renewal traffic

$$(29) \qquad z_{ov} > \min(1,z).$$

In this context we note that $z_{ov}$ may be smaller than z as shown by Pearce [21]; also, $z_{ov}$ may be smaller than 1 (see [23]).

We finally remark that the example in Whitt [29] referred to in the previous section, can be used to show that (25) (and hence (26)) is not generally valid for non-renewal traffic. A closer look reveals the following. One can derive the representation

$$(30) \qquad z_{ov} - (1-B+Bz) = (1-B)(M_{ov}^{ov}-M_{ov}^{ca}) + M'_{ov} - BM',$$

where $M'_{ov}$ ($M_{ov}^{ca}$) is the expected number of busy trunks on the overflow group at an arrival moment of the offered stream (at a moment where a call is accepted on the finite group). For renewal traffic one now has $M'_{ov} = BM'$ (by Lemma 1) and $M_{ov}^{ov} \geq M_{ov}^{ca}$ (by Theorem 4), but neither result holds in general for non-renewal traffic as appears from Whitt's example.

## 5. ADDITIONAL RESULTS ON RENEWAL TRAFFIC

In this section we will collect some results for the peakedness factor of renewal traffic, most of which are folklore. As usual we assume the offered traffic to be free and stationary (the latter amounts to considering equilibrium renewal processes in the terminology of Cox [5]) and holding times to be mutually independent, exponentially distributed random variables with a common mean $1/\mu$. Let $F(t)$, with $F(0+) = 0$, be the distribution function of the interarrival times and let

$$\phi(s) = \int_0^\infty e^{-st} dF(t), \quad \text{Re } s \geq 0,$$

be its Laplace-Stieltjes transform. The intensity of the arrival stream is now given by $\lambda$ where

$$(31) \qquad \lambda^{-1} = \int_0^\infty t dF(t) = -\phi'(0).$$

Regarding the trunk occupancy distribution on an infinite trunk group, offered this renewal traffic, we clearly have $M = A = \lambda/\mu$. As for $M'$, let $\tau$ be the length of an interarrival interval and let $n^+$ ($n^-$) denote the number of busy trunks just after it has started (just before it ends). Then, with $E$ denoting expectation,

$$M' = E\{n^-\} = E_\tau\{E_{n^+}\{E_{n^-}\{n^-|n^+,\tau\}\}\} =$$

$$E_\tau\{E_{n^+}\{n^+e^{-\mu\tau}|\tau\}\} = E_{n^+}\{n^+\}E_\tau\{e^{-\mu\tau}\}.$$

Thus $M' = \phi(\mu)E\{n^+\}$. Since $E\{n^+\} = 1 + M'$, it follows that

(32) $$M' = \phi(\mu)/(1-\phi(\mu)).$$

Consequently, (6) yields

(33) $$z = \frac{1}{1-\phi(\mu)} - \frac{\lambda}{\mu},$$

which is a well-known result.

REMARK. One can obtain (32) also from Cohen [3] or Theorem 5 of Takács [27]. Alternatively, one can use Theorem 6 of Takács [27] and the definition of z to obtain (33) directly, considering that the condition which Takács imposes on F (of being not a lattice distribution) is dictated only by his considering limiting distributions instead of stationary distributions.

Two types of renewal traffic deserve special mentioning because of their frequent occurrence in teletraffic theory.

1. *Hypo-exponential traffic* is defined by the relation

$$F = F_1 * E_2 * \dots * E_k,$$

where * denotes convolution and $E_i$, $i = 1,2,\dots,k$, an exponential distribution with mean $1/\lambda_i$, such that

(34) $$\sum_i 1/\lambda_i = 1/\lambda.$$

Thus an interarrival time may be thought of as consisting of k independent, exponentially distributed phases. Clearly, we have

(35) $$\phi(s) = \prod_i (\lambda_i/(\lambda_i+s)),$$

whence the peakedness factor is given by

(36) $$z = \frac{\Pi(1+\lambda_i/\mu)}{\Pi(1+\lambda_i/\mu)-\Pi(\lambda_i/\mu)} - \frac{\lambda}{\mu}.$$

Considering that $u(x) = x/(x-c)$ is decreasing in x and

(37) $$\prod_i (1+\lambda_i/\mu) \geq \prod_i (\lambda_i/\mu) + \sum_j \prod_{i\neq j} (\lambda_i/\mu),$$

we can use (37) in both numerator and denominator of (36) to conclude that $z \leq 1$, with equality subsisting only if $k = 1$ (exponential traffic, more commonly called *Poisson traffic*).

If the k phases have equal means $(k\lambda)^{-1}$ we speak of *Erlang traffic*. For $k \to \infty$ the interarrival time distribution of Erlang traffic tends to a degenerate distribution $F_R(t)$ which concentrates all mass at $1/\lambda$ and corresponds to *regular traffic*. For this particular case we have

(38) $$\phi_R(s) = \exp(-s/\lambda),$$

whence

(39) $$z_R = \{1-\exp(-\mu/\lambda)\}^{-1} - \lambda/\mu,$$

a well-known result that we have used in Section 2.

2. *Hyper-exponential traffic* has an interarrival distribution of the form

$$F = \sum_1^k a_i E_i,$$

where $E_i$, $i = 1,2,\ldots,k$, is an exponential distribution with mean $1/\lambda_i$ and

(40) $$a_i > 0,\ \sum_i a_i = 1,\ \sum_i a_i/\lambda_i = 1/\lambda.$$

Evidently, the Laplace-Stieltjes transform of F is

(41) $$\phi(s) = \sum_i a_i\lambda_i/(\lambda_i+s),$$

so that the peakedness factor of hyperexponential traffic is given by

(42) $$z = \{\sum_i a_i/(1+\lambda_i/\mu)\}^{-1} - \lambda/\mu.$$

Considering that $u(x) = x/(1+x)$ is strictly concave, the mean of $\{u(\mu/\lambda_i)\}_i$ weighted by the $a_i$'s is smaller than the value of u in the point $\Sigma_i a_i\mu/\lambda_i = \mu/\lambda$. That is, for $k > 1$,

$$\frac{\mu}{\lambda}/(1+\frac{\mu}{\lambda}) > \sum_i a_i \frac{\mu}{\lambda_i}/(1+\frac{\mu}{\lambda_i}).$$

Hence $z \geq 1$, with equality subsisting only if $k = 1$ (Poisson traffic).

In conclusion we mention that Palm [20] (see also Wallin [28], and for more general results Van Doorn [7]) has shown that traffic which is overflowing from a finite number N of trunks offered Poisson traffic is hyper-exponential with $k = N + 1$, whence this type of traffic is peaked. The latter result follows of course directly from Pearce's inequality (29).

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